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Thierry Paul

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Rossby waves trapped by quantum mechanics

THIERRY PAUL

Rossby and Poincaré waves appear naturally in the study of large scale oceanography. Poincaré waves (PW), of period of the order of a day, are fast dispersive waves and are due to the rotation of the Earth through the Coriolis force. Much slower, Rossby waves (RW) are sensitive to the variations of the Coriolis parameter, propagate only eastwards and remain localized for long period of times. We would like here to report on some new results, obtained in collaboration with C. Cheverry, I. Gallagher and L. Saint-Raymond [1, 2, 3] studying this phenomenon, dispersivity of PW and trapping of RW, as a consequence of the study of the oceanic waves in a shallow water flow subject to strong wind forcing and rotation, linearized around a inhomogeneous (non zonal) stationary profile. The main feature of our results, compared to earlier ones, [5, 7, 4] to quote only very few of them, consists in the fact that we abandon both the betaplane approximation (constant Coriolis force) and the zonal aspect (non dependence w.r.t. the latitude) of the convection term (coupling with the wind).

After some scalings and dimensional homogenizations, the Saint-Venant system of equations for the variations η, u near a constant value of the height \bar{h} and divergence free stationary profile of velocity \bar{u} takes the form (see [2, 3] for details)

$$(1) \quad \begin{aligned} \partial_t \eta + \frac{1}{\varepsilon} \nabla \cdot u + \bar{u} \cdot \nabla \eta + \varepsilon^2 \nabla \cdot (\eta u) &= 0 \\ \partial_t u + \frac{1}{\varepsilon^2} b u^\perp + \frac{1}{\varepsilon} \nabla \eta + \bar{u} \cdot \nabla u + u \cdot \nabla \bar{u} + \varepsilon^2 u \cdot \nabla u &= 0 \end{aligned}$$

where b is the horizontal component of the Earth rotation vector normalized to one and ε^{-1} measures the Coriolis force.

The linear version of (1) reads (here $D := \frac{1}{i} \partial$ and $x = (x_1, x_2) \in \mathbb{R}^2$):

$$(2) \quad \varepsilon^2 i \partial_t \mathbf{v} + A(x, \varepsilon D, \varepsilon) \mathbf{v} = 0, \quad \mathbf{v} = (v_0, v_1, v_2) = (\eta, u_1, u_2),$$

with the linear propagator

$$(3) \quad A(x, \varepsilon D, \varepsilon) := i \begin{pmatrix} \varepsilon \bar{u} \cdot \varepsilon \nabla & \varepsilon \partial_1 & \varepsilon \partial_2 \\ \varepsilon \partial_1 & \varepsilon \bar{u} \cdot \varepsilon \nabla + \varepsilon^2 \partial_1 \bar{u}_1 & -b + \varepsilon^2 \partial_2 \bar{u}_1 \\ \varepsilon \partial_2 & b + \varepsilon^2 \partial_1 \bar{u}_2 & \varepsilon \bar{u} \cdot \varepsilon \nabla + \varepsilon^2 \partial_2 \bar{u}_2 \end{pmatrix}.$$

We will concentrate on (2) with the condition that, essentially, b is increasing at infinity with all derivatives bounded in module by $|b|$ and only non degenerate critical points. Moreover \bar{u} will have to be smooth with compact support.

A simplified version of our main result reads as follows (see [3] for details).

Theorem 1. *Under certain microlocalization properties of the initial condition, the solution $\mathbf{v}_\varepsilon(t) = \mathbf{v}_\varepsilon(t, \cdot)$ of (2) decomposes on two Rossby and Poincaré vector fields $\mathbf{v}_\varepsilon(t) = \mathbf{v}_\varepsilon^R(t) + \mathbf{v}_\varepsilon^P(t)$ satisfying*

- $\forall t > 0, \forall \Omega$ compact set of \mathbb{R}^2 ,

$$(4) \quad \|\mathbf{v}_\varepsilon^P(t)\|_{L^2(\Omega)} = O(\varepsilon^\infty)$$

- $\exists \Omega$ bounded set of \mathbb{R} such that, $\forall t > 0$

$$(5) \quad \|\mathbf{v}_\varepsilon^R(t)\|_{L^2(\mathbb{R}_{x_1} \times (\mathbb{R} \setminus \Omega)_{x_2})} = O(\varepsilon^\infty).$$

Theorem 1 shows clearly the different nature of the two type of waves: dispersion for Poincaré and confining in x_2 for Rossby. The method of proving Theorem 1 will consist in diagonalizing the “matrix” $A(x, \varepsilon D, \varepsilon)$. Such diagonalization, if possible, would immediately solve (2) by reducing it to the form $\varepsilon^2 i \partial_t \mathbf{u} + D(x, \varepsilon D, \varepsilon) \mathbf{u} = 0$ with $D(x, \varepsilon D, \varepsilon)$ diagonal and solving it component by component. Diagonalizing matrices with operator valued entries is not a simple task, but our next result will show how to achieve it modulo ε^∞ in the case of matrices with ε -semiclassical type operators entries.

To any (regular enough) function $\mathcal{A}_\varepsilon \sim \sum_0^\infty \varepsilon^l \mathcal{A}_l$ on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, possibly matrix valued, we associate the operator A_ε (densely defined) on $L^2(\mathbb{R}^n)$ defined by:

$$f \rightarrow A_\varepsilon f, (A_\varepsilon f)(x) = \int \mathcal{A}_\varepsilon\left(\frac{x+y}{2}, \xi\right) e^{i \frac{\xi(x-y)}{\varepsilon}} f(y) \frac{dy d\xi}{\varepsilon^n}.$$

\mathcal{A}_ε is called the symbol of A_ε and A_ε the (Weyl) quantization of \mathcal{A}_ε .

Let A_ε be such $N \times N$ operator valued matrix of symbol $\mathcal{A}_\varepsilon \sim \sum_0^\infty \varepsilon^l \mathcal{A}_l$. We will suppose that $\mathcal{A}_0(x, \xi)$ is Hermitian and therefore is diagonalizable (at each point) by $\mathcal{U} = \mathcal{U}(x, \xi)$, $\mathcal{U}^* \mathcal{A}_0 \mathcal{U} = \text{diag}(\lambda_1, \dots, \lambda_N) := \mathcal{D}$. We will suppose moreover that

$$(6) \quad \forall (x, \xi), \quad \forall i \neq j, \quad |\lambda_i(x, \xi) - \lambda_j(x, \xi)| \geq C > 0.$$

Theorem 2 ([3]). *There exist V_ε semiclassical operator and D_ε diagonal (w.r.t. the $N \times N$ structure) such that*

$$V_\varepsilon^{-1} A_\varepsilon V_\varepsilon = D_\varepsilon + O(\varepsilon^\infty) \quad \text{and} \quad V_\varepsilon^* V_\varepsilon = Id_{L^2(\mathbb{R}^n, \mathbb{C}^N)} + O(\varepsilon^\infty) = V_\varepsilon V_\varepsilon^* + O(\varepsilon^\infty).$$

Moreover $D_\varepsilon = D + \varepsilon D_1 + O(\varepsilon^2)$, where D is the Weyl quantization of \mathcal{D} and D_1 is the diagonal part of $(\Delta_1 - \frac{D I_1 + I_1 D}{2})$ with $(U$ being the Weyl quantization of \mathcal{U})

$$(7) \quad \Delta_1 = \frac{U^* A_\varepsilon U - D}{\varepsilon} \Big|_{\varepsilon=0}, \quad I_1 = \frac{U^* U - Id_{L^2(\mathbb{R}^n, \mathbb{C}^N)}}{\varepsilon} \Big|_{\varepsilon=0}.$$

Let us go back now to the case given by (3). One checks easily that $A(x, \varepsilon D, \varepsilon)$ is of semiclassical type. Its symbol is

$$(8) \quad \mathcal{A}(x, \xi, \varepsilon) = \begin{pmatrix} \varepsilon \bar{u} \cdot \xi & \xi_1 & \xi_2 \\ \xi_1 & \varepsilon \bar{u} \cdot \xi + \varepsilon^2 \partial_1 \bar{u}_1 & -b + \varepsilon^2 \partial_2 \bar{u}_1 \\ \xi_2 & b + \varepsilon^2 \partial_1 \bar{u}_2 & \varepsilon \bar{u} \cdot \xi + \varepsilon^2 \partial_2 \bar{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & -b \\ \xi_2 & b & 0 \end{pmatrix} + O(\varepsilon).$$

The spectrum of the leading order $\mathcal{A}(x, \xi, 0)$ is $\{-\sqrt{\xi^2 + b^2(x_2)}, 0, +\sqrt{\xi^2 + b^2(x_2)}\}$. Therefore Condition (6) is satisfied only if $\xi^2 + b^2(x) \geq C > 0$ which correspond to the microlocalization condition in Theorem 1. Theorem 2 gives, after a tedious computation, that $A(x, \varepsilon D, \varepsilon)$ is unitary equivalent (modulo ε^2) to the diagonal matrix $\text{diag}(T^+, T^R, T^-)$ where T^\pm is the Weyl quantization of $\tau^\pm(x, \xi) := \pm \sqrt{\xi^2 + b^2(x_2)}$ and T^R is the quantization of the Rossby Hamiltonian $\tau^R(x, \xi) := \varepsilon \left(\frac{\xi_1 b'(x_2)}{\xi^2 + b^2(x)} + \bar{u}(x) \cdot \xi \right)$.

Under the betaplane approximation, $b(x_2) = \beta x_2$, the Hamiltonians T^\pm are exactly solvable and one shows by hand the dispersive effect for the Poincaré waves. In our situation this doesn't work, and because of the ε^2 term in the r.h.s. of (2) the method of characteristics does not apply. A general argument, inherited from quantum mechanics will provide us the solution. First we remark that the Poisson bracket $\{\tau^\pm, x_1\} = \xi_1 / \tau^\pm$. This indicates, at a classical level, that \dot{x}_1 has

a sign for each Poincaré polarization, leading to no return travel. The following theory, due to Eric Mourre, gives the “quantum” equivalent of this argument.

Let H and A be two self-adjoint operators on a Hilbert space \mathcal{H} such that: the intersection of the domains of H and A is dense in the domain of H , $t \mapsto e^{itA}$ maps the domain of H to itself and $\sup_{[0,1]} \|He^{itA}\varphi\| < \infty$ for φ in the domain of H , and $i[H, A]$ is bounded from below, closable and the domain of its closure contains the domain of H . Finally let us suppose the following

Positivity condition: there exist $\theta > 0$ and an open interval Δ of \mathbb{R} such that if E_Δ is the corresponding spectral projection of H , then

$$(9) \quad E_\Delta i[H, A] E_\Delta \geq \theta E_\Delta,$$

namely $i[H, A] > 0$ on any spectral interval of H contained in Δ .

Theorem 3 (E. Mourre '80, [6]). *For any integer $m \in \mathbb{N}$ and for any $\theta' \in]0, \theta[$, there is a constant C such that*

$$\|\chi_-(A - a - \theta't)e^{-iHt}g(H)\chi_+(A - a)\| \leq Ct^{-m}$$

where χ_\pm is the characteristic function of \mathbb{R}^\pm , g is any smooth compactly supported function in Δ , and the above bound is uniform in $a \in \mathbb{R}$.

In other words, to talk in the quantum language, if one starts with an initial condition φ such that “ $A \geq a$ ” and the positivity condition (9) holds, after any time t the “probability” that “ $A \leq \theta't$ ” is of order t^{-m} . In particular, as $t \rightarrow \infty$ the solution $e^{-iHt}\varphi$ escape from any compact spectral region of A .

Taking $A = x_1$, Theorem 3 gives, after verification that it applies, exactly the “Poincaré” part of Theorem 1. The “Rossby part” is given by using the bicharacteristic method and a small computation done in [3] which shows that bicharacteristics are trapped in finite regions in the latitude (x_2) direction.

Let us mention to finish that the nonlinear terms can be handled by using a “ L^∞ ” Gronwall Lemma and working in some anisotropic and semiclassical Sobolev spaces, so that the solution of (1) is close to the one of (2) as $\varepsilon \rightarrow 0$.

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